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ON THE WEDDERBURN PRINCIPAL THEOREM FOR NEARLY (1, 1) ALGEBRAS

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Abstract. A nearly (1, 1) algebra is a finite dimensional strictly power-associative algebra satisfying the identity $(x, x, y) = (x, y, x)$ where the associator $(x, y, z) = (xy)z - x(yz)$. An algebra A has a Wedderburn decomposition in case A has a subalgebra $S \cong A - N$ with $A = S + N$ (vector space direct sum) where N denotes the radical (maximal nil ideal) of A.

D. J. Rodabaugh has shown that certain classes of nearly (1, 1) algebras have Wedderburn decompositions. The object of this paper is to expand these classes. The main result is that a nearly (1, 1) algebra A containing 1 over a splitting field of characteristic not 2 or 3 such that A has no nodal subalgebras has a Wedderburn decomposition.

Introduction. An algebra A has a Wedderburn decomposition in case A has a subalgebra $S \cong A - N$ with $A = S + N$ (vector space direct sum) where N denotes the radical of A. A class of algebras is called a Wedderburn class provided that each algebra in the class has a Wedderburn decomposition. These include associative [1], alternative [9], commutative power-associative [4], Jordan [2], [7], [10, pp. 106f], and other algebras [8]. The Wedderburn principal theorem for a class C of algebras states that if an algebra A in C has the property that $A - N$ is separable, then A has a Wedderburn decomposition.

In this paper, an algebra A is a finite dimensional vector space on which a multiplication is defined that satisfies both distributive laws and the condition that $\alpha(xy) = (\alpha x)y = x(\alpha y)$ for x, y in A and α in the field. Define $x^1 = x$ and $x^{k+1} = x^k x$ for every x in A and every positive integer k. A power-associative algebra A is one for which $x^{k+m} = x^k x^m$ for every x in A and all positive integers k and m. If A_K is power-associative for every scalar extension K of the base field, then A is called strictly power-associative. The radical N of A is the maximal nil ideal of A, i.e., the maximal ideal of A consisting entirely of nilpotent elements. For x, y, and z in A, the associator $(x, y, z) = (xy)z - x(yz)$.

A power-associative algebra A whose base field has characteristic not 2 with an idempotent e ($e^2 = e \neq 0$) has a Peirce decomposition $A = A_1(e) + A_{1/2}(e) + A_0(e)$

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where $A_i(e)=\{x \in A : ex=xe=ix\}$ for $i=0$, 1 and $A_{1/2}(e)=\{x \in A : ex+xe=x\}$ [3, p. 560]. The subset $A_i(e)$ for $i=0, 1/2, 1$ is also denoted by $A(e, i)$ or, when unambiguous, by A_i. An idempotent e of an algebra A is called a primitive idempotent in case e is the only idempotent in $A_1(e)$. This idempotent e is called absolutely primitive provided e is primitive in A_K for every extension K of the base field of A.

A field K is a splitting field of an algebra A if and only if every primitive idempotent e of A_K-N_K is absolutely primitive and every element in $(A_K-N_K)(e, 1)$ for e primitive can be written as $\alpha e+y$ with α in K and y nilpotent.

An algebra is nodal provided each element can be written as $\alpha 1+z$ with α in the base field and z nilpotent where the set of nilpotent elements is not a subalgebra.

A nearly (1, 1) algebra is defined to be a strictly power-associative algebra that satisfies the identity

$$(x, x, y) = (x, y, x). \tag{1}$$

Nearly (1, 1) algebras were first studied by Kleinfeld, Kosier, Osborn, and Rodabaugh [6] as a special type of associator dependent algebras. A nearly $(-1, 0)$ algebra is an algebra that is anti-isomorphic to a nearly (1, 1) algebra. The nearly (1, 1) and nearly $(-1, 0)$ algebras are generalizations of (1, 1) and $(-1, 0)$ algebras respectively. The (1, 1) and $(-1, 0)$ algebras are particular types of (γ, δ) algebras. The properties of nearly (1, 1) algebras listed in the remainder of this paragraph have been proved by them in [6]. A nearly (1, 1) algebra over a field of characteristic not 2 or 3 has a Peirce decomposition

$$A = A_{11}+A_{10}+A_{01}+A_{00}$$

where $A_{ij}=\{x \in A : ex=ix \text{ and } xe=jx\}$ for $i, j=0, 1$. Also, the subspaces A_{ij} satisfy the following relations where $i=0$ or 1 and $j=1-i$:

(2) $A_{ii}^2 \subset A_{ii}$,
(3) $A_{ii}A_{jj}=0$,
(4) $A_{ij}^2 \subset A_{ji}$,
(5) $x_{ij}^2=0$ where x_{ij} is in A_{ij},
(6) $A_{ij}A_{ji} \subset A_{ii}$,
(7) $A_{ij}A_{jj} \subset A_{ij}$,
(8) $A_{jj}A_{ij}=0$,
(9) $A_{ii}A_{ij} \subset A_{ij}+A_{jj}$,
(10) $A_{ij}A_{ii} \subset A_{jj}$,
(11) $x_{ii}y_{ij}-y_{ij}x_{ii}$ is in A_{ij} when x_{ii} is in A_{ii} and y_{ij} is in A_{ij}.

Defining $G_i=A_{ji}A_{jj}$ for $i=0$ or 1 and $j=1-i$, then $G=G_1+G_0$ is an ideal of A with $G^2=0$.

Rodabaugh has shown [8, Theorem 6.2] that a (1, 1) or $(-1, 0)$ algebra over a splitting field of characteristic not 2 or 3 has a Wedderburn decomposition. Also, he has shown [8, Theorems 6.1 and 6.3] that if A is a nearly (1, 1) (nearly $(-1, 0)$)

algebra over a splitting field of characteristic not 2 or 3 such that either (a) $A-N$ is associative where $N=G(e)$ for each idempotent $e\neq 1$ in A or (b) A contains neither nodal subalgebras nor ideals K with $K^2=0$, then A has a Wedderburn decomposition. In this paper it is shown that several classes of nearly (1, 1) (nearly $(-1, 0)$) algebras are Wedderburn classes. The main result is that a nearly (1, 1) (nearly $(-1, 0)$) algebra over a splitting field of characteristic not 2 or 3 with no nodal subalgebras has a Wedderburn decomposition.

Main section. We first prove that under rather restrictive conditions a nearly (1, 1) algebra contains a Cayley subalgebra. This result is used to show that some nearly (1, 1) algebras have Wedderburn decompositions. This result, in turn, is extended in Theorems 2, 3, and 4.

Linearizing (1) by replacing x with $x+z$ gives the identity

$$(x, z, y)+(z, x, y)-(x, y, z)-(z, y, x) = 0 \tag{12}$$

in a nearly (1, 1) algebra. Partially linearize $(x, x, x)=0$ by replacing x by $x+y$ to obtain $(x, x, y)+(x, y, x)+(y, x, x)=0$. This together with (1) implies

$$2(x, x, y)+(y, x, x) = 0 \tag{13}$$

in a nearly (1, 1) algebra.

THEOREM 1. *Let A be a nearly* (1, 1) *algebra containing* 1 *over a base field F of characteristic not* 2 *or* 3 *with $N=G(e)$ for each idempotent $e\neq 1$ in A. Suppose $A-N$ is a split Cayley algebra over F. Then A contains a split Cayley subalgebra.*

Proof. Since $A-N$ is a split Cayley algebra over F, we may suppose that $A-N=M_2+[w]M_2$ where M_2 is the algebra of all two-by-two matrices over F [10, Lemma 3.16] and $[w]$ indicates the image of w in the natural mapping $A\to A-N$. Furthermore,

$$[w]^2 = [1] \tag{14}$$

and multiplication in $A-N$ is given by

$$a([w]b) = [w](\bar{a}b), \tag{15}$$

$$([w]a)b = [w](ba), \tag{16}$$

$$([w]a)([w]b) = b\bar{a} \tag{17}$$

for a, b in M_2 where for $c=\alpha[u_{11}]+\beta[u_{12}]+\gamma[u_{21}]+\delta[u_{22}]$ in M_2 with $\alpha, \beta, \gamma, \delta$ in F and $\{[u_{ij}]\}_{i,j=1,2}$ the set of matric units for M_2, $\bar{c}=\alpha[u_{22}]-\beta[u_{12}]-\gamma[u_{21}]+\delta[u_{11}]$ [10, Chapter III, §4].

Rodabaugh has shown [8, Proof of Lemma 6.1] that there exists a subalgebra B of A such that $B\cong M_2$ and there exists a basis $\{e_{ij}\}_{i,j=1,2}$ of B with e_{11} and e_{22} idempotents such that $1=e_{11}+e_{22}$. Also, e_{ij} is in $B_{10}(e_{ii})\cap B_{01}(e_{ij})$ for $i\neq j$; $i, j=1, 2$. Furthermore, $[e_{ij}]=[u_{ij}]$ for $i, j=1, 2$. Also

$$e_{ij}e_{km} = \delta_{jk}e_{im} \quad \text{for } i, j, k, m = 1, 2 \tag{18}$$

where δ_{jk} is the Kronecker delta. Let $[f_{12}]=[w][e_{22}]$ and $[f_{21}]=[w][e_{11}]$. Consider the 8 elements e_{ii}, e_{ij}, f_{ij}, $f_{ij}e_{ij}$ for $i, j=1, 2$ and $i\neq j$. Then $[e_{ii}]=[u_{ii}]$, $[e_{ij}]=[u_{ij}]$, $[f_{ij}]=[w][u_{jj}]$, and $[f_{ij}e_{ij}]=[w][u_{ij}]$ using (15) and (18). Since $\{[e_{ii}], [e_{ij}], [f_{ij}], [f_{ij}e_{ij}]\}$ is a basis of $A-N$, $\{e_{ii}, e_{ij}, f_{ij}, f_{ij}e_{ij}\}$ is a basis for an 8 dimensional subspace C of A. We now show that C is a subalgebra of A isomorphic to $A-N$ under the natural mapping $A\to A-N$ restricted to C. We do this by examining the multiplication of basis elements of C.

Since $w=w_{11}+w_{10}+w_{01}+w_{00}$ where w_{ij} is in $A_{ij}(e_{11})$ for $i, j=0, 1$, it follows that $[f_{12}]=[w_{10}]+[w_{00}]$ and $[f_{21}]=[w_{01}]+[w_{11}]$. Now $[w_{10}]=[e_{11}]([w_{10}]+[w_{00}])$ $=[e_{11}][f_{12}]=[e_{11}]([w][e_{22}])=[w]([e_{11}]^{-}[e_{22}])=[w][e_{22}]=[f_{12}]=[w_{10}]+[w_{00}]$ using (15). Thus, $[w_{00}]=[0]$, so $[f_{12}]=[w_{10}]$, so we may choose f_{12} in $A_{10}(e_{11})$. Also,

$$[w_{11}] = [e_{11}]([w_{01}]+[w_{11}]) = [e_{11}][f_{21}] = [e_{11}]([w][e_{11}]) = [w]([e_{11}]^{-}[e_{11}]) = [0]$$

using (15). Thus, $[f_{21}]=[w_{01}]$, so f_{21} may be chosen in $A_{01}(e_{11})$. It is convenient notationally to let $A_{12}=A_{10}$, $A_{21}=A_{01}$, and $A_{22}=A_{00}$, i.e., we replace the subscript 0 with the subscript 2. In the remainder of this proof A_{ij} denotes $A_{ij}(e_{11})$ for $i, j=1, 2$. We have just chosen f_{ij} in A_{ij} for $i\neq j$; $i, j=1, 2$.

Using (17), we have $[f_{ij}f_{ji}]=[e_{ii}]$, so $f_{ij}f_{ji}-e_{ii}$ is in $N=G(e_{11})$. By (6) $f_{ij}f_{ji}$ is in A_{ii}, so

$$a_i = f_{ij}f_{ji}-e_{ii} \text{ is in } N\cap A_{ii}. \tag{19}$$

Using (11) and the fact N is an ideal, we have $a_if_{ij}-f_{ij}a_i$ is in $N\cap A_{ij}=G(e_{11})\cap A_{ij}$ $=(A_{21}A_{22}+A_{12}A_{11})\cap A_{ij}\subset(A_{11}+A_{22})\cap A_{ij}=0$ by (10), so

$$a_if_{ij} = f_{ij}a_i. \tag{20}$$

By (8),

$$a_if_{ji} = 0. \tag{21}$$

From (13) with $x=f_{ij}$ and $y=f_{ji}$, (5), (19), and (21), we have

$$f_{ij}a_i = 0. \tag{22}$$

This with (20) implies

$$a_if_{ij} = 0. \tag{23}$$

Using (12) with $x=f_{ij}$, $y=a_i$, and $z=f_{ji}$, (19), (22), (21), (23), and the facts that a_i is in $N\cap A_{ii}$ and $N^2=[G(e_{11})]^2=0$, we have $a_i-f_{ij}(f_{ji}a_i)-(f_{ji}a_i)f_{ij}=0$, so $a_i-f_{ij}(f_{ji}a_i)=(f_{ji}a_i)f_{ij}$ is in $A_{ii}\cap A_{jj}=0$ by (19), (7), and (6). Consequently,

$$a_i = f_{ij}(f_{ji}a_i). \tag{24}$$

Using (12) with $x=f_{ij}$, $y=f_{ji}$, and $z=a_i$, (22), (21), (23), (19), (24), and the fact that a_i is in $N\cap A_{ii}$ with $N^2=0$, we have $a_i=0$, so by (19)

$$f_{ij}f_{ji} = e_{ii}. \tag{25}$$

In (12) let $x=e_{ii}$, $y=e_{ij}$, and $z=f_{ij}$, then use the fact that f_{ij} is in A_{ij}, (4), and (18) to get

$$e_{ij}f_{ij} = -f_{ij}e_{ij}. \tag{26}$$

Next we wish to show that $e_{ij}f_{ji}=0=f_{ij}e_{ji}$. If a_{ij} and b_{ij} are in A_{ij}, then $0=(a_{ij}+b_{ij})^2=a_{ij}b_{ij}+b_{ij}a_{ij}$, so

$$a_{ij}b_{ij} = -b_{ij}a_{ij}. \tag{27}$$

Let $c_j=e_{ji}f_{ij}$ for $j=1, 2$ and $i\neq j$. By (6), c_j is in A_{jj}. With the aid of (15), we have $[c_j]=[0]$ so c_j is in N. Thus,

$$c_j \text{ is in } A_{jj} \cap N. \tag{28}$$

Using (11), we have $c_ie_{ij}-e_{ij}c_i$ is in $A_{ij} \cap N=0$. This together with (10) implies

$$c_ie_{ij} = e_{ij}c_i \quad \text{in } A_{jj}. \tag{29}$$

From (12) with $x=c_j$, $y=e_{ji}$, and $z=e_{ij}$, (28), (9), (18) and (29), we have $c_j=(c_je_{ji})e_{ij}-(e_{ij}c_j)e_{ji}$. Properties (28) and (7) imply $e_{ij}c_j$ is in $A_{ij} \cap N=0$. Thus, $c_j=(c_je_{ji})e_{ij}$, so with (29) we know

$$c_j = (e_{ji}c_j)e_{ij}. \tag{30}$$

In (13) let $x=e_{ji}$ and $y=f_{ij}$ to get $e_{ji}c_j=0$ with the help of (5), (6), and (8). This with (30) implies $c_j=0$ or

$$e_{ji}f_{ij} = 0. \tag{31}$$

Let $d_i=f_{ij}e_{ji}$. It follows, using (16), that $[d_i]=[0]$, so d_i is in N. This together with (6) implies

$$d_i \text{ is in } A_{ii} \cap N. \tag{32}$$

Using (11), we have $d_je_{ji}-e_{ji}d_j$ is in $A_{ji} \cap N=0$ so from (32) and (10) we obtain

$$d_je_{ji} = e_{ji}d_j \text{ is in } A_{ii} \cap N. \tag{33}$$

In (12) let $x=d_j$, $y=e_{ji}$, and $z=e_{ij}$, then employ (32), (8), (18), (32) again, and (33) to obtain $d_j=(d_je_{ji})e_{ij}-(e_{ij}d_j)e_{ji}$. Using (32) and (7), we have $e_{ij}d_j$ is in $A_{ij} \cap N=0$, so

$$d_j = (d_je_{ji})e_{ij}. \tag{34}$$

Utilizing (12) with $x=f_{ij}$, $y=e_{ij}$, and $z=e_{ji}$, (18), (31), and (27), we get $d_ie_{ij}=f_{ij}+2e_{ji}(f_{ij}e_{ij})-(f_{ij}e_{ij})e_{ji}$. This with (33) and (4) implies d_ie_{ij} is in $A_{jj} \cap A_{ij}=0$, so from (34) $d_i=0$ or

$$f_{ij}e_{ji} = 0. \tag{35}$$

By (13) with $x=f_{ij}$ and $y=e_{ij}$, (5), (4), and (6), we have $(e_{ij}f_{ij})f_{ij}=2f_{ij}(f_{ij}e_{ij})$ is in $A_{jj} \cap A_{ii}=0$, so

$$(e_{ij}f_{ij})f_{ij} = 0. \tag{36}$$

Equations (27) and (36) imply

$$(37) \qquad (f_{ij}e_{ij})f_{ij} = 0.$$

Equation (36) and the line preceding it imply

$$(38) \qquad f_{ij}(f_{ij}e_{ij}) = 0.$$

Using (16), we have

$$(39) \qquad [f_{ij}e_{ij}] = [w][e_{ij}].$$

We obtain upon employing (39) and (17) $[f_{ji}(f_{ij}e_{ij})]=[e_{ij}]$. This with (4) shows that $f_{ji}(f_{ij}e_{ij})-e_{ij}$ is in $N \cap A_{ij}=0$, so

$$(40) \qquad f_{ji}(f_{ij}e_{ij}) = e_{ij}.$$

Equations (27) and (40) imply

$$(41) \qquad (f_{ij}e_{ij})f_{ji} = -e_{ij}.$$

Utilizing (13) with $x=e_{ij}$ and $y=f_{ij}$, (5), (4), and (6), we get $(f_{ij}e_{ij})e_{ij}=2e_{ij}(e_{ij}f_{ij})$ is in $A_{jj} \cap A_{ii}=0$, so

$$(42) \qquad (f_{ij}e_{ij})e_{ij} = 0,$$

and $e_{ij}(e_{ij}f_{ij})=0$ which with (27) gives

$$(43) \qquad e_{ij}(f_{ij}e_{ij}) = 0.$$

From (39) and (16) we have $[(f_{ji}e_{ji})e_{ij}]=[f_{ji}]$, so with the aid of (4) we get $(f_{ji}e_{ji})e_{ij}-f_{ji}$ is in $N \cap A_{ji}=0$, so

$$(44) \qquad (f_{ji}e_{ji})e_{ij} = f_{ji}.$$

This with (4) and (27) implies

$$(45) \qquad e_{ij}(f_{ji}e_{ji}) = -f_{ji}.$$

Let $g_{ji}=f_{ij}e_{ij}$ which is in A_{ji} by (4). Using (12) with $x=g_{ji}$, $y=f_{ji}$, and $z=e_{ji}$, (27), (4), and (6), we obtain $2g_{ji}(f_{ji}e_{ji})+2e_{ji}(f_{ji}g_{ji})=(g_{ji}f_{ji})e_{ji}+(e_{ji}f_{ji})g_{ji}$ is in $A_{jj} \cap A_{ii}$ $=0$, so $g_{ji}(f_{ji}e_{ji})=-e_{ji}(f_{ji}g_{ji})$ which with (40) and (18) yields

$$(46) \qquad (f_{ij}e_{ij})(f_{ji}e_{ji}) = -e_{jj}.$$

Notice that multiplication of basis elements of C is (isomorphically) the same as multiplication of basis elements of $A-N$ from the fact that f_{ij} is in A_{ij}, (5), (18), (25), (26), (31), (35), (37), (38), and (40)–(46). Thus C is a split Cayley algebra.

COROLLARY. *Suppose A is a nearly* (1, 1) *algebra containing* 1 *over a base field F of characteristic not* 2 *or* 3 *such that $N=G(e)$ for each idempotent $e\neq 1$ in A. Also, suppose $A-N$ is a split Cayley algebra. Then A has a Wedderburn decomposition.*

Proof. By the preceding theorem, A contains a split Cayley algebra C. Since there is a unique split Cayley algebra over F, $C \cong A-N$. The radical of $A-N$ is 0, so the radical of C is 0. But $N \cap C$ is a nil ideal of C, so $N \cap C=0$. Then a dimension argument shows that $A=C+N$, a Wedderburn decomposition of A.

We now strengthen this corollary to the following

THEOREM 2. *Let A be a nearly* (1, 1) (*nearly* (−1, 0)) *algebra containing* 1 *over a base field F of characteristic not* 2 *or* 3. *Suppose $A-N$ is a split Cayley algebra over F. Then A has a Wedderburn decomposition.*

Proof. Let Q be the class of algebras B over F satisfying the hypotheses of this theorem, i.e., (i) B is nearly (1, 1), (ii) 1 is in B, and (iii) $B-N_B$ is a split Cayley algebra where N_B denotes the radical of B. Let A be in Q. The proof proceeds by induction on the dimension of A. Since $A-N$ is a Cayley algebra, $\dim (A-N)=8$, so $\dim A \geqq 8$. If A has dimension 8, then $A \cong A-N$. Since $\operatorname{rad}(A-N)=0$, $N=\operatorname{rad} A=0$, so $A-N=A-0 \cong A$, implying $A=A+0$ is a Wedderburn decomposition of A. Suppose $\dim A=n>8$ and assume inductively that every algebra B in Q having dimension less than n has a Wedderburn decomposition. By Lemma 2.2 of [8], A has a Wedderburn decomposition if it can be shown that A contains an ideal M other than 0, N, and A and that Q has the properties: (a) if B is in Q, then $B-N_B$ is simple; (b) if B is in Q and $M \subset N_B$ is an ideal of B, then $B-M$ is in Q; and (c) if B is in Q and C is a subalgebra of B whose image in $B \to B-N_B$ is a nonnil ideal of $B-N_B$, then C is in Q. With heavy reliance on the isomorphism theorems one can show that Q has properties (a), (b), and (c).

If $N=G(e)$ for every idempotent $e \neq 1$ in A, then by the corollary to Theorem 1, A has a Wedderburn decomposition. Note that A contains an idempotent different from 1. Since $A-N$ is a split Cayley algebra, $A-N$ contains [an isomorphic copy of] M_2, so $A-N$ contains the matric unit $[u_{11}]$. By [8, Lemma 2.1], A contains an idempotent e such that $[e]=[u_{11}]$. If $e=1$, then $[1]=[e]=[u_{11}]$ contrary to the definition of $[u_{11}]$. Thus A contains an idempotent different from 1, namely e. This proof will be complete when we treat the possibility that A has an idempotent $e \neq 1$ such that $N \neq G(e)$.

Suppose A contains such an idempotent e. If the ideal $G(e) \neq 0, A$, then by Lemma 2.2 of [8], A has a Wedderburn decomposition. We know $G(e) \neq A$ since 1 is in A^2 but $(G(e))^2=0$. Suppose $G(e)=0$. Then $0=G(e)=G_1(e)+G_0(e)=A_{01}A_{00}+A_{10}A_{11}$ where for the remainder of this proof A_{ij} denotes $A_{ij}(e)$ for $i, j=0, 1$. Thus,

$$A_{ij}A_{ii} = 0 \quad \text{for } i = 0 \text{ or } 1 \text{ and } j = 1-i \text{ when } G = 0. \tag{47}$$

Let a_{11} be in A_{11} and a_{10} be in A_{10}. In (12) let $x=e$, $y=a_{11}$, and $z=a_{10}$ and apply (47) to get $-a_{11}a_{10}+e(a_{11}a_{10})=0$. It follows, using (9), that $A_{11}A_{10} \subset A_{10}$. Utilizing (12) with $x=a_{00}$ in A_{00}, $y=a_{01}$ in A_{01}, and $z=e$, (47), and (9), we have $e(a_{00}a_{01})=0$, so $a_{00}a_{01}$ is in $A_{00}+A_{01}$. By (9), $a_{00}a_{01}$ is in $A_{01}+A_{11}$. Thus, $a_{00}a_{01}$ is in A_{01}, or $A_{00}A_{01} \subset A_{01}$. We now have

$$A_{ii}A_{ij} \subset A_{ij} \quad \text{for } i = 0 \text{ or } 1 \text{ and } j = 1-i \text{ when } G = 0. \tag{48}$$

Let $L = A_{10}A_{01} + A_{10} + A_{01} + A_{01}A_{10}$. From the proof of Lemma 4 of [6] together with (47) and (48) it follows that L is an ideal of A. If $L \neq 0$, N, A, then A has a Wedderburn decomposition by Lemma 2.2 of [8]. Consider, then, the three remaining cases.

Case 1. Suppose $L=0$. Then $A_{10}=0=A_{01}$, so $A=A_{11}+A_{00}$. Now A_{11} is an ideal of A, $A_{11}\neq 0$ since e is in A_{11}, $A_{11}\neq N$ since e is in A_{11} but is not nilpotent, and $A_{11}\neq A$ since $e\neq 1$. By Lemma 2.2 of [8], A has a Wedderburn decomposition.

Case 2. Suppose $L=N$. Then $A_{01}\subset N$ and $A_{10}\subset N$, so taking a Peirce decomposition of $A-N$ with respect to $[e]$, we have $A-N=(A-N)_{00}+(A-N)_{11}$. Thus $(A-N)_{00}$ is an ideal of $A-N$. But $A-N$ is a split Cayley algebra, so is simple, so $(A-N)_{00}=0$ or $A-N$. Since $[1]$ is in $A-N$ but not $(A-N)_{00}$, $(A-N)_{00}=0$. Hence $A-N=(A-N)_{11}$. Then $[e]$ is the identity of $A-N$ so $[e]=[1]$ implying that $e-1$ is nilpotent. However, $e-1\neq 0$ since $e\neq 1$, $(e-1)^2=e^2-2e+1=-e+1\neq 0$, and inductively $(e-1)^n=(-1)^{n+1}(e-1)\neq 0$, for any positive integer n. This contradiction forces us to discard this case, i.e., $L\neq N$.

Case 3. Suppose $L=A$. Then $A_{10}A_{01}+A_{10}+A_{01}+A_{01}A_{10}=A_{11}+A_{10}+A_{01}+A_{00}$, so $A_{10}A_{01}=A_{11}$ and $A_{01}A_{10}=A_{00}$. By Lemma 5 of [6], A_{11} and A_{00} are associative (hence alternative) subrings of A. From the proof of Theorem 4 of [6] together with (47) and (48), A is alternative. Thus by [10, Theorem 3.18], A has a Wedderburn decomposition.

We have now shown that A must have a Wedderburn decomposition. By mathematical induction, any algebra A in Q has a Wedderburn decomposition. A nearly $(-1, 0)$ algebra satisfying the hypotheses of this theorem is anti-isomorphic to a nearly $(1, 1)$ algebra satisfying the hypotheses of this theorem. Since the latter has a Wedderburn decomposition, the former does also.

We now modify Theorem 2 to the following result.

THEOREM 3. *If A is a nearly* $(1, 1)$ (*nearly* $(-1, 0)$) *algebra containing* 1 *over a base field F of characteristic not* 2 *or* 3 *such that $A-N$ is a Cayley algebra and $N^2=0$, then A has a Wedderburn decomposition.*

Proof. Since $A-N$ is a Cayley algebra, [9, p. 605] states that there is a scalar extension K of finite degree over F such that $(A-N)_K$ is a split Cayley algebra. We show that the radical R of A_K is N_K. $R-N_K=\{[x] : x \text{ is in } R\}$, where $[x]$ denotes the image of x in the natural mapping $A_K \to A_K-N_K$, is an ideal of the simple algebra $(A-N)_K$, so $R-N_K=0$ or $R-N_K=(A-N)_K$. But $R-N_K\neq(A-N)_K$ since $[1]$ is in $(A-N)_K$ but not $R-N_K$. Thus $R-N_K=0$, so $R\subset N_K$. Since $N^2=0$, $N_K^2=0$, hence N_K is a nil ideal of A_K, so $N_K\subset R$. Therefore, $R=N_K$. The remainder of the proof is the same as the associative case as given by Albert in Theorem 3.23 of [1].

We now prove the main result of this paper.

THEOREM 4. *If A is a nearly* $(1, 1)$ (*nearly* $(-1, 0)$) *algebra over a splitting field of characteristic not* 2 *or* 3 *such that A has no nodal subalgebras, then A has a Wedderburn decomposition.*

Proof. Let P be the class of all algebras A satisfying the hypotheses of this theorem, i.e., A is in P provided

(i) A is an algebra over a splitting field of characteristic not 2 or 3,

(ii) A is nearly (1, 1),

(iii) A contains no nodal subalgebras.

First, we show that P is a decomposable class as defined by Rodabaugh in [8], i.e., for each A in P

(a) A is strictly power-associative over a field of characteristic not 2 or 3,

(b) $A-N$ is in P,

(c) if B is a subalgebra of A whose image in $A \to A-N$ is a nonnil ideal in $A-N$, then B is in P,

(d) if A is semisimple (A nonnil and $N=0$), then $A=A_1 \oplus\cdots\oplus A_t$ where each A_i is simple with a unity element, and

(e) $A_t(e)A_t(e) \subset A_t(e)$ for $t=0$, 1 if e is an idempotent in A.

Let A be in P with base field F. By (i) and (ii), condition (a) is satisfied. $A-N$ is in P since (i) F is a splitting field of $A-N$, (ii) $A-N$ is nearly (1, 1), and (iii) by Theorem 4.2 of [8], $A-N$ contains no nodal subalgebras since A contains none. Thus, (b) is satisfied. Suppose B is a subalgebra of A whose image in T: $A \to A-N$ is a nonnil ideal in $A-N$. It follows that F is a splitting field of $T(B)$, of $B-\text{rad } B$, and of B, so B satisfies (i). Clearly B, being a subalgebra of A, satisfies (ii) and (iii). Thus, B is in P, so (c) is satisfied. By Theorems 7 and 9 of [6], (d) is satisfied. By (2), (e) is satisfied. Thus, P is a decomposable class.

By Theorem 2.1 of [8], P is a Wedderburn class if the center $C(P)$ of P is, where a member A of P is in $C(P)$ provided 1 is in A and $A-N$ is simple. Let A be in $C(P)$. To show that A has a Wedderburn decomposition, we consider two cases according to whether 1 is the only idempotent of A or whether A has an idempotent different from 1.

Case 1. Suppose 1 is the only idempotent of A. Then [1] is a primitive idempotent of $A-N$ since if $[f]$ is an idempotent of $(A-N)([1], 1)$, then there exists an idempotent e in A such that $[e]=[f]$ by Lemma 2.1 of [8], but $e=1$, so $[f]=[1]$. Since F is a splitting field of $A-N$, every element in $A-N=(A-N)([1], 1)$ can be written as $\alpha[1]+[y]$ with α in F and $[y]$ nilpotent. If the set of nilpotent elements of $A-N$ does not form a subalgebra of $A-N$, then $A-N$ is nodal, so A is nodal by Theorem 4.2 of [8] contrary to the hypothesis of this theorem. Thus, the set B of nilpotent elements of $A-N$ is a subalgebra of $A-N$. Furthermore, B is an ideal of $A-N$. Since A is in $C(P)$, $A-N$ is simple, so $B=0$ or $B=A-N$. However, [1] is in $A-N$ but not B, so $B=0$. Thus $A-N=\{\alpha[1] : \alpha \text{ is in } F\}$. Clearly $A-N \cong F1$, so $A=F1+N$ is a Wedderburn decomposition of A.

Case 2. Suppose A has an idempotent $e \neq 1$. Then $[e] \neq [1]$. By Theorem 6 of [6], $A-N$ is alternative. Since [1] is in $A-N$, $A-N$ is nonnil. Kleinfeld [5] has shown that a simple nonnil alternative algebra is either a Cayley algebra or is associative. Consequently, $A-N$ is either a Cayley algebra or is associative.

Suppose $A-N$ is a Cayley algebra. If $A-N$ were a division algebra, then $[e]([e]-[1]) = [e]^2-[e]=[0]$, so $[e]=[0]$ or $[e]=[1]$ contrary to e being an idempotent different from 1. Thus, $A-N$ is a split Cayley algebra. By Theorem 2, A has a Wedderburn decomposition. If $A-N$ is associative, A has a Wedderburn decomposition by Theorem 6.1 of [8].

We have now shown that $C(P)$ is a Wedderburn class, so P is also. Thus every nearly (1, 1) algebra satisfying the hypotheses of this theorem has a Wedderburn decomposition. A nearly $(-1, 0)$ algebra satisfying the hypotheses of this theorem is anti-isomorphic to a nearly (1, 1) algebra satisfying these hypotheses so has a Wedderburn decomposition.

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